The THEORY *of* FRUSTUMS

and its justification of

EULER'S LEMMA

and application to
the solution of

$$X^3 \pm Y^3 \neq Z^3$$

Ernest C. Johnson

To; R.D. Carmichael,

with compliments of

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Introduction

In order that the significance of the following treatise may be more fully appreciated, I am including in this introduction an outline of the history of the problems which it concerns, and a short sketch of my own efforts toward the solution of these problems.

Diophantus, the celebrated mathematician of ancient Greece, noted in some of his writings that no two cubes have ever been found whose sum equals a cube. Just how long before that time, the problem of the cubes originated, is not known, but since that time there are many records of attempts to find two cubes whose sum equals a cube, and many more records of attempts to prove that no such numbers exist.

Fermat, the eminent French mathematician who lived about four hundred years ago, and who made valuable advances in the science of mathematics, devoted a part of his time to the study of this problem. After his death a note was found on the margin of one of his books, to the effect that he had discovered a truly remarkable proof that the sum of two numbers raised to any power greater than the square cannot equal another number raised to the same power, but he further stated that the lack of space prevented giving the proof there. Mathematicians, to this present day, have tried without success to supply the missing proof to this problem which has become known among them as "Fermat's Problem." One enthusiast bequeathed one hundred thousand marks as a prize which stands at the disposal of the University of Gottingen, for an acceptable proof of Fermat's Problem.

Realizing that a solution to the problem of the cubes would be a valuable approach to the solution of Fermat's Problem, many of the most famous mathematicians directed their attention toward a proof of that problem first. The most notable of all attempts to prove that $x^3+y^3=z^3$ cannot be satisfied by whole numbers, was one made by Leonhard Euler, a Swiss mathematician, during the early

part of the Eighteenth Century. His proof, which is given in the latter part of this book, contained two assumptions for which he had no rigorous proof. Because of these assumptions he was later forced to admit that he had failed to give a complete proof of the problem. His approach to the solution, however, seemed so logical that other mathematicians tried to supply the required proof of his assumptions. Thus, "The Completion of Euler's proof" became, in itself, a notable problem, and for two centuries has defied the efforts of many of the world's best mathematicians.

My own efforts toward the solution of these problems began during the summer of 1914 when I first heard of the one concerning the cubes. Without knowing anything of their history or the difficulties which had been encountered by others who sought their solution, I attempted, at odd times and in spare moments, to solve the problem of the cubes. I approached the problem from almost every conceivable angle, and nearly every attempt ended with some proposition concerning a quanity of the form (x^2+xy+y^2). The most prominent of the propositions required a proof that the cube root of (x^2+xy+y^2) is a number of like form. Not being able to supply this proof, I temporarily abandoned any further work on the approaches which led up to that point. Then one day I came across a short sketch of "Euler's Proof" in which I found that he stated without proof that p^2+3q^2 is a cube of t^2+3u^2 of like form. I further saw that his (p^2+3q^2) may be made to equal my (x^2+xy+y^2) and that Euler had, therefore, been unable to give a proof for the very proposition which had stopped me.

I could plainly see that the completion of my own work from this point, as well as the filling in of the gap left by Euler, depended upon a fuller understanding of the properties of numbers which may be expressed as (x^2+xy+y^2). I resolved, at once, to make a systematic study of those numbers, and to keep a record of my findings. In as much as I did this work at odd times when not engaged in my regular work, my progress was very slow, so that it was more than a year later (September 26, 1927,

to be exact) that I made my first decided advance by completing my formula for expressing the product of two numbers of the form (x^2+xy+y^2) in that same form. It was then more than another year before I was able to prove that every factor of a number of the form (x^2+xy+y^2) may be expressed in that form.

Upon the completion of this last proposition I was able to take up my original work where I had left off, and prove that the cube root of a number of the form (x^2+xy+y^2) is a number of like form. I was also, upon the completion of this last proposition, able to prove, for the first time in more than two hundred years, that Euler's first assumption was correct.

By this time another serious complication had developed to oppose further progress. I had found that the cube of a number of the form (x^2+xy+y^2) may be expressed in that same form in many different ways. The number of these expressions for a single cube varies so greatly that for some large numbers they run into hundreds or even thousands of variations. Every one of these different expressions of the cube must be accounted for and considered in the proper solution of the problem. It was in this connection that Euler made his second assumption, that $p+q\sqrt{-3}$ must be the cube of $t+u\sqrt{-3}$, and was criticised by other mathematicians since $t+u\sqrt{-3}$ might also have other expressions of its cube.

After considerable labor I found that all these different expressions of a cube may be grouped together into a few classes. All but one of these classes were eliminated from further consideration as being irrelevant to the permissible assumption that x and y be relatively prime and both odd. The solution was then finally completed by showing that the remaining class of expressions of a cube of that form do not satisfy the condition that x and y be the smallest integers which will satisfy the given equation $x^3+y^3=z^3$.

By a similar process I was able to show that Euler's $p+q\sqrt{-3}$ is the only cube of $t+u\sqrt{-3}$ which will satisfy his assumption that x and y are relatively prime and both

odd, and that the other expressions of the cube which he did not consider but which were pointed out by his critics, really do not need any further consideration because they do not satisfy his permissible assumption that x and y are relatively prime and both odd.

I have therefore provided a proof that the sum of two cubes cannot equal a cube, and completed Euler's Proof by showng that his two assumptions are correct.

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Definitions

For the purpose of avoiding lengthy repetitions, I have, in the presentation of this work, used the following terms which are defined as follows:

Frustum, is an algebraic expression of the form (x^2+xy+y^2), or any number which may be expressed in that form.

Base. The terms used to express a frustum will be known as the "First Base," "Middle Base," and "Last Base" respectively. That is, in the Frustum (x^2+xy+y^2), x^2 will be known as the "First Base," xy the "Middle Base," and y^2 the "Last Base."

Base Root. The square root of the first base or last base, will be known as a "Base Root." That is, x and y will be known as the "base roots" of the frustum (x^2+xy+y^2).

Cycle. The six sets of base roots which may express any given Frustum, because of the manner in which they are related to each other, will be known collectively as a cycle. (see proposition 11).

Development of the Theory of Frustums

Proposition 1, Theorem. A frustum may be written in reversed order without changing its value.

That is, (x^2+xy+y^2) may be written (y^2+yx+x^2) without changing its value, for $x^2+xy+y^2=y^2+yx+x^2$.

Proposition 2, Theorem. If the base roots of a frustum are relatively prime, and their difference a multiple of three, the frustum is a multiple of three but not a multiple of nine.

Given; a frustum (x^2+xy+y^2) with x and y relatively prime and $(x-y)$ a multiple of three.

To show that the frustum is a multiple of three but not a multiple of nine.

$(x^2+xy+y^2)=(x-y)^2+3xy$, Hence if $(x-y)$ is a multiple of three, $(x-y)^2$ must be some multiple of nine. But x and y, being relatively prime, neither can be a multiple of three, and $3xy$ cannot be a multiple of nine. Hence $(x-y)^2+3xy$ is a multiple of three but not a multiple of nine.

Proposition 3, Problem. To express the product of two given frustums as a frustum.

Given (a^2+ab+b^2) and (x^2+xy+y^2).

To express their product as a frustum, make the first base root of the product equal to the product of the sums of the base roots of the given frustums, minus the pro-

duct of the first base roots of the given frustums; and make the last base root of the product equal to the product of the first base roots of the given frustums, minus the product of the last base roots of the given frustums, and write it, $(ay+bx+by)^2+(ay+bx+by)(ax-by)+(ax-by)^2$, for $(a^2+ab+b^2)\times(x^2+xy+y^2)=a^2x^2+a^2xy+a^2y^2+abx^2+abxy+aby^2+b^2x^2+b^2xy+b^2y^2$, which equals $[(a+b)(x+y)-ax]^2+[(a+b)(x+y)-ax]\times(ax-by)+(ax-by)^2$ as per the above formula, and which may be written, $(ay+bx+by)^2+(ay+bx+by)(ax-by)+(ax-by)^2$.*

Proposition 4, Problem. To express the product of two frustums, whose bases are positive, by two different sets of positive bases.

Given; two frustums with positive bases, (a^2+ab+b^2) and (x^2+xy+y^2).

To express their product by two different sets of positive bases.

Arrange the bases of the given frustums so that a is greater than b, and x is greater than y, and obtain the first expression of their product by multiplying $(a^2+ab+b^2)\times(x^2+xy+y^2)$ which equals $(ay+bx+by)^2+(ay+bx+by)\times(ax-by)+(ax-by)^2$, both bases of which are positive. The first, because all its signs are positive, and the last because ax is greater than by.

Then obtain the second expression of their product by reversing the frustum whose first base divided by its last base is the least, and multiplying as before.

*NOTE:—There are also several other ways of combining the given frustums, but in the work which follows, "the product of two frustums," or any other expression signifying one frustum multiplied by another, will always mean that the product is to be expressed as per the above formula, unless specifically stated to be otherwise.

Suppose $\frac{a}{b}$ is less than $\frac{x}{y}$, then reverse the frustum whose base roots are a and b,and multiply $(b^2+ba+a^2)\times(x^2+xy+y^2)$ which equals $(ax+ay+by)^2+(ax+ay+by)(bx-ay)+(bx-ay)^2$, both bases of which are positive. The first because all its signs are positive, and the last because the inequality $\left(\frac{a}{b}<\frac{x}{y}\right)$ multiplied by (by) equals $(ay<bx)$, and therefore $bx-ay$ is positive.

Proposition 5, Problem. To arrange the base roots of two different sets of positive bases which express the same frustum, in such a manner that the difference of the base roots of one set, plus the difference of the base roots of the other set, will equal some positive multiple of three.

Given; $(a^2+ab+b^2)=(x^2+xy+y^2)$.

To arrange a, b, x, and y, so that $a-b+x-y=$some positive multiple of three.

Arrange the frustums so that a is greater than b, and x greater than y. Then, if (a^2+ab+b^2) and (x^2+xy+y^2) are multiples of three, $(a-b+x-y)$ equals some positive multiple of three, for $(a^2+ab+b^2)-3ab$, and $(x^2+xy+y^2)-3xy$, and therefore, their square roots, $a-b$, and $x-y$, would each be positive multiples of three. Also the sum of those square roots, $(a-b+x-y)$, would be a positive multiple of three.

However, if (a^2+ab+b^2) and (x^2+xy+y^2) be, some (multiple of three)+1, then $(a^2-2ab+b^2)$, and $(x^2-2xy+y^2)$, are each a (multiple of 3) +1, but their square roots, $a-b$, and $x-y$, might be either a (multiple of 3) +1, or a (multiple of 3) −1. But if $(a-b)$ is a (multiple of 3)+1, and $(a-y)$ some (multiple of 3)−1, or vice versa, their sum $(a-b+x-y)$ would be some positive multiple of three.

In the case when $(a-b)$ and $(x-y)$ are the same with respect to being a (multiple of 3) ± 1, reverse the frustum in which the difference of the base roots is least, for if $(a-b)$ is less than $(x-y)$, and the frustum (a^2+ab+b^2) be reversed so that $a=b_1$, and $b=a_1$, then a_1-b_1+x-y will equal some positive multiple of three.

Proposition 6, Problem. To find two factors, expressed as frustums with positive integral base roots, whose product is equal to a given frustum which may be expressed by two different given sets of positive integral base roots.

Given; $(a^2+ab+b^2)=(x^2+xy+y^2)$, in which a, b, x, and y, are positive integers. To find two factors, (m^2+mn+n^2), and (t^2+tu+u^2), whose product is equal to (a^2+ab+b^2), or (x^2+xy+y^2), with m, n, t, and u, being positive integers.

Arrange a, b, x, and y, as per proposition 5, so that $a-b+x-y$ is some multiple of 3.

Now, let the greatest common divisor of $(a-b+x-y)$, and $(a+2b+x+2y)$ equal $3(m+n)$, and

$$\frac{a-b+x-y}{3(m+n)}=u, \text{ and } \frac{a+2b+x+2y}{3(m+n)}=t,$$

Now, having the value if t, and u, let $\dfrac{a-b+x+2y}{3(t+u)}=n$, then $(m+n)-n=m$.

To derive the above formula, and prove that m, n, t, and u, are positive integers, let (a^2+ab+b^2) express the product of (m^2+mn+n^2) and (t^2+tu+u^2), taken together in that order, which equals $(tn+um+un)^2+(tn+um+un)(tm-un)+(tm-un)^2$, and let (x^2+xy+y^2) express the product of the same frustums, taken together in the order $(n^2+nm+m)^2\times(t^2+tu+u^2)$, as per proposition 4, and written $(tm+um+un)^2+(tm+um+un)(tn-um)+(tn-um)^2$, so that $a=tn+um+un$, $b=tm-un$, $x=tm+um+un$, and $y=tn-um$.

By addition and subtraction, obtain the following equations;

$$(1.)\quad a-b+x-y=3u(m+n),$$

$$(2.)\quad a+2b+x+2y=3t(m+n),$$

$$(3.)\quad a-b+x+2y=3n(t+u),$$

As a, b, x, and y, are given integers, the above equations are each integers. Then if $3(m+n)$ be made to equal the greatest common integral divisor of equations (1) and (2), u and t will each be integers and relatively prime. From equation (3), $n=\frac{a-b+x+2y}{3(t+u)}$, hence if n is not an integer it must be a fraction whose denominator is some factor of $(t+u)$.

But from, $a=tn+u(m+n)$, $n=\frac{a-u(m+n)}{t}$, hence n must be a fraction whose denominator is some factor of t as well as a factor of $(t+u)$, but as t is prime to u, and therefore prime to $(t+u)$, they have no common factor except unity, hence n, and therefore m, must be an integer.

Proposition 7, Theorem. A prime frustum may be expressed by only one set of positive integral base roots.

Proof; If a frustum may be expressed by more than one set of positive integral base roots it may be factored as per Proposition 6, and is therefore not prime. Hence a prime frustum, which can not be factored, cannot be expressed by more than one set of positive integral base roots.

Proposition 8, Theorem. If from the base roots of a given frustum a common factor be removed, the remaining base roots will express a frustum which is equal to the given frustum divided by the square of the common factor.

Given; (a^2+ab+b^2), and a factor c, common to a and b.

Now if the base roots a and b be divided by c, the resulting base roots $\frac{a}{c}$ and $\frac{b}{c}$ will express the frustum $\frac{a^2}{c^2}+\frac{ab}{c^2}+\frac{b^2}{c^2}$ which is equal to $\frac{(a^2+ab+b^2)}{c^2}$, the given frustum divided by the common factor squared.

Proposition 9, Theorem. If the product of two frustums be multiplied by one of those frustums without reversing the order of its bases, both base roots of the last product will be multiples of that frustum.

Given; the frustum

$$(ay+bx+by)^2+(ay+bx+by)(ax-by)+(ax-by)^2$$

which is equal to the product of

$$(a^2+ab+b^2) \text{ and } (x^2+xy+y^2).$$

To show that the product of the given frustum and its factor (x^2+xy+y^2) will be a frutum whose base roots are even multiples of (x^2+xy+y^2).

Multiply them together as per proposition 3, their product will be $(ay^2+ax^2+axy)^2+(ay^2+ax^2+axy)$ $(bx^2+bxy+by^2)+(bx^2+bxy+by^2)^2$

Whose 1st base root is $a(x^2+xy+y^2)$

and last base root $b(x^2+xy+y^2)$

each of which is seen to be multiples of the frustum (x^2+xy+y^2).

Proposition 10, Problem. To divide a given composite frustum by a given known frustum factor, and express the quotient as a frustum.

Given; a composite frustum (a^2+ab+b^2), and another frustum (m^2+mn+n^2) which is known to be a factor of (a^2+ab+b^2).

To express their quotient as a frustum, multiply (a^2+ab+b^2) by (m^2+mn+n^2)*, and divide each base root of their product by (m^2+mn+n^2), the quotients thus obtained will be the two base roots of the required frustum.

Proof; The product of (a^2+ab+b^2) and (m^2+mn+n^2)* will be a frustum whose base roots are both divisible by (m^2+mn+n^2), proposition 9.

And if each of those base roots be divided by (m^2+mn+n^2), the quotents thus obtained will be base roots of a frustum which is equal to $\dfrac{(a^2+ab+b^2)(m^2+mn+n^2)}{(m^2+mn+n^2)^2}$, as proposition 8, and that is equal to $\dfrac{(a^2+ab+b^2)}{(m^2+mn+n^2)}$ the required quotient, which may be written as a frusum

$$\frac{(an+bm+bn)^2}{(m^2+mn+n^2)}+\frac{(an+bm+bn)(am-bn)}{(m^2+mn+n^2)}+\frac{(am-bn)^2}{(m^2+mn+n^2)}$$

Proposition 11, Theorem. A frustum may be expressed, in terms of its own base roots, by six different sets of base roots, functionally related in a cycle, in such a manner that the first base root of any set is equal to the sum of the base roots of the previous set, and the last base root of any set is equal to minus the first base root of the previous set.

Example;

(1.) $x^2 \quad +xy \quad +y^2$
(2.) $(x+y)^2+(x+y)(-x)+(-x)^2$
(3.) $y^2+y[-(x+y)]+[-(x+y)]^2$
(4.) $(-x)^2+(-x)(-y)+(-y)^2$
(5.) $[-(x+y)]^2+[-(x+y)]x+x^2$
(6.) $(-y)^2+(-y)(x+y)+(x+y)^2$

*NOTE:—In some cases it might be necessary to reverse the factor (m^2+mn+n^2) before multiplying and dividing, in order to make the base roots of the required quotient come out in integers, because proposition 9 says "if the product of two frustums be multiplied by one of those frustums, **without reversing** the order of its bases, both base roots of the last product will be multiples of that frustum." Hence if the original product had been obtained by using the known factor in reversed order of that first tried, it would be necessary to use the factor in that same order in solving the above problem.

In the foregoing list of frustums, each with a separate set of base roots, but expressing the same frustum, any given set of base roots in the list may be derived from the previous set, as per the theorem. In the same manner (1) may be derived from (6) thus continuing and repeating the same order. Hence the sets are said to be related in a cycle.

To prove that the various sets of base roots listed above all express the same frustum, let a and b be made to equal the first and last base roots, respectively, of any given set, so that the frustum equal (a^2+ab+b^2). Then the next set derived from it becomes $[(a+b)^2+(a+b)(-a)+(-a)^2]$ which, when cleared of parentheses is $[a^2+2ab+b^2-a^2-ab+a^2]$ and equals (a^2+ab+b^2). In that manner any set may be shown to be equal to the previous set and, therefore, to each set in the cycle.

Definition; Standard Cycle, shall be understood to mean the six sets of base roots which express the same frustum in terms of its own base roots, arranged in such a manner that the first member of the cycle shall always be expressed by positive base roots, and the following members of the cycle numbered and arranged in order of their relation to each other as per the above example.

Proposition 12, Theorem. If the two base roots of a frustum contain a common divisor, each set of base roots in the cycle, of which this frustum is a member, will contain that same common divisor.

Given; the frustum (x^2+xy+y^2), in which x and y, have a common divisor d.

To show that each set of base roots in the cycle of which (x^2+xy+y^2) is a member, contains that same common divisor.

Let $x=ad$, and $y=bd$, so that the frustum (x^2+xy+y^2) may be written, $(ad)^2+(ad)(bd)+(bd)^2$. Now the next arrangement in the cycle, as per proposition 11, is $(ad+bd)^2+(ad+bd)(-ad)+(-ad)^2$, in which each base root is divisible

by (d), In that same manner each following arrangement in a cycle may be shown to contain the same common divisor as its preceding arrangement, and thus each set of base roots in the cycle contains the same common divisor.

Conversely; If the two base roots of a frustum are relatively prime the base roots of each set in the cycle of which this frustum is a member, will be relatively prime, for, if the base roots of any set are not relatively prime, they have a common divisor, and, as per the above theorem, each set of base roots in the cycle will contain that same common divisor, which is contrary to our assumption that two of the bases are relatively prime.

Proposition 13, Theorem. The standard cycle contains all possible arrangements of signs.

Proof; The number of possible arrangements of two signs taken two at a time when each may appear twice is, according to the law of permutations, 2^2 or 4.

It can be verified by inspection that the standard cycle contains four different arrangements of signs, viz; both base roots positive; both base roots negative; first base root positive and last base root negative; and first base root negative and last base root positive; which is all the possible arrangements of signs.

Proposition 14, Theorem. The Standard Cycle contains all possible arrangements of base roots which may express a given frustum in terms of a given set of base roots.

Proof; Suppose there is some other arrangement of base roots not listed in the standard cycle (proposition 11) which may express the frustum in terms of the given base roots. The signs of this supposed arrangement must, at least, correspond with some arrangement of signs in the standard cycle, for the standard cycle contains all possible arrangements of signs (proposition 13). Now, if we build

a cycle from this supposed arrangement, the next arrangement must correspond to the next arrangement in the standard cycle, etc, etc, until the cycle is complete. One of these arrangements must have both positive base roots, for in one arrangement of the standard cycle both are positive (proposition 13). Now if the above supposition were possible, we might easily substitute the base roots of a prime frustum for the given set of base roots and thereby obtain two sets of positive integral base roots for a prime frustum, but that is impossible, for a prime frustum may be expressed by only one set of positive integral base roots (proposition 7). Hence it is not possible that there could be another set of base roots expressing the given frustum in terms of the given base roots, which does not appear in the standard cycle, and therefore, the standard cycle contains all possible arrangements of base roots which may express a given frustum in terms of any given set of base roots.*

Proposition 15, Theorem. Of the six sets of base roots expressing a frustum in terms of its own base roots, one and only one set may be expressed by all positive base roots,

Proof; Since the standard cycle contains all possible arrangements of signs, (theorem 13) and all possible arrangements of base roots which may express a frustum in terms of its own base roots (theorem 14), it follows that each one of the six sets of base roots in any cycle must have a corresponding set in the standard cycle, but the standard cycle contains one set with all positive base roots, hence any cycle must contain one set with all positive base roots, and as the other five sets must correspond each to each with the other five sets in the standard cycle there cannot be another set with all positive base roots.

*NOTE:—Each arrangement in the cycle may be reversed (proposition 1), making in all, twelve different arrangements of the base roots expressing a frustum in terms of a given set of base roots.

Proposition 16, Theorem. In the product of two frustums, an identical arrangement of base roots may be obtained by six different combinations of the various arrangements of base roots in the cycles of the two frustums.

Given; two frustums (a^2+ab+b^2) and (x^2+xy+y^2) whose product is written,

$$(ay+bx+by)^2+(ay+bx+by)(ax-by)+(ax-by)^2,$$

To obtain another expression of the frustums (a^2+ab+b^2) and (x^2+xy+y^2), whose product, written as per proposition 3, will be

$$(ay+bx+by)^2+(ay+bx+by)(ax-by)+(ax-by)^2,$$

pass one step backward through the cycle of the one frustum and one step forward through the cycle of the other frustum and write their product as per propsition 3. That is,

$[(a+b)^2+(a+b)(-a)+(-a)^2]\times[(-y)^2+(-y)(x+y)+(x+y)^2]$ (as per proposition 3), equals

$$(ay+bx+by)^2+(ay+bx+by)(ax-by)+(ax-by)^2,$$

which is exactly the same as the given product. In that manner we may pass through each entire cycle of six arrangements, making six different combinations of base roots whose products are all written the same.

Proposition 17, Theorem. The product of two given frustums may be expressed as a frustum, in terms of the base roots of the given frustums, by only two different cycles, and therefore, only two sets of positive integral base roots.

Given; two frustums (a^2+ab+b^2) and (x^2+xy+y^2). To show that their product may be expressed by only two different cycles, or two set of positive integral base roots, in terms of a, b, x, and y.

Proof; Each frustum may be expressed in

terms of its own base roots by twelve and only twelve different arrangements, (note on proposition 14), hence the product of the two frustums may be obtained in twelve times twelve, or one hundred forty-four, different ways. But an identical arrangement of base roots may be obtained by six different arrangements (proposition 16). Hence there must be only twenty-four different arrangements of base roots in the product. Each arrangement of base roots will also appear in reversed order, (proposition 14) leaving only twelve really different sets of base roots, or two cycles of six sets each, (proposition 14) and two sets of positive integral base roots, (proposition 15).

Proposition 18, Theorem. In the product of several prime frustums, the order in which they are combined makes no difference in the identity or number of possible sets of base roots which may be obtained to express their product as a frustum.

Given; Any number of prime frustums, multiplied together in the order $[(1^{st} \times 2^{nd}) \times 3^{rd}] \cdots \times n^{th}$, having made, at each operation, all possible sets of base roots which may express that product as a frustum.

Now, if the final product may be expressed by some other set of base roots not obtainable by that paticular order of multiplication, and if that expression of the product be divided by the n^{th} factor, in the manner used in theorem 10, then the quotient, thus obtained, must differ from all sets of base roots expressing the product of the first $n-1$ factors, for if it corresponds to one of these sets of base roots, then its product, when multiplied by the n^{th} factor, would correspond with one of the sets of base roots expressing the final product as a frustum, but that is contrary to our assumption.

For the same reason, the set of base roots obtained by dividing that quotient by the $(n-1)^{th}$, factor, will differ from all sets of base roots which express the product of the first $n-2$ factors, and so on down through, each successive quotient obtained by dividing, successively by the

several factors in the reversed order of which they were multiplied, will differ from all of the several sets of base roots expressing their products, until finally the quotient obtained by dividing by the 3rd factor, must differ from those sets of base roots obtained by the product of the 1st and 2nd factors. But that is contrary to the assumption that they are combined to form all possible set of base roots which may express their product. Hence, the final product of any given number of prime frustums cannot be expressed by any set of base roots not obtainable through that order of combining them. Therefore, the order in which they are combined, makes no difference in the identity or number of possible sets of base roots which may be obtained to express their product as a frustum.

Proposition 19, Theorem. A composite frustum which is equal to the product of n prime frustums, may be expressed in terms of the base roots of these n prime frustums, by just 2^{n-1} sets of positive bases and cycles.

Proof; Let r be the number of different sets of positive base roots which express the product of n prime frustums in terms of their base roots.

Now, if each of these r frustums be multiplied by another prime frustum, making two sets of positive bases expressing each product as per proposition 4, there would then be $2r$ sets of positive base roots expressing the product of $n+1$ prime frustums. Since each additional prime frustum doubles the number of expressions of the product, it holds that for $n-1$ prime furstums there must be $\frac{r}{2}$ different sets of positive bases, etc, etc., to $[n-(n-2)]$ prime frustums expressed by $\frac{r}{2^{n-2}}$ different sets. But the product of two frustums is expressed by two sets of positive bases (proposition 17). Hence $\frac{r}{2^{n-2}}=2$, or $r=2^{n-1}$, equals the

number of sets of positive bases, and cycles, which may express the product of n prime frustums in terms of their base roots.

Proposition 20, Theorem. Every factor of a composit frustum may be expressed as a frustum.

Given; Any composite frustum expressed as (x^2-xy+y^2), arranged so that x is greater than y, and resolved into two factors, at least one of which may be expressed as $(x-a)$, for since (x^2-xy+y^2) is less than (x^2), at least one of its factors must be less than x.

To show that $(x-a)$, and $[(x^2-xy+y^2)\div(x-a)]$ may be expressed as frustums.

By division,

$$(x^2-xy+y^2)\div(x-a)=x-y+a+\frac{a^2-ay+y^2.}{x-a}$$

Now since $x-y+a+\dfrac{a^2-ay+y^2}{x-a}$ must be a whole number, $\dfrac{a^2-ay+y^2}{x-a}$ must be a whole number, hence a^2-ay+y^2 is either equal to, or a multiple of, $x-a$ if it is equal to $x-a$, then $(x-a)$ may be expressed as a frustum a^2-ay+y^2, but if it is some multiple of $(x-a)$ then it may be divided in the following manner; if a is greater than $(x-a)$, let $(a-b)=(x-a)$ and divide as before, making

$$(a^2-ay+y^2)\div(a-b)=a-y+b+\frac{b^2-by+y^2}{a-b}$$

in which, as before, b^2-by+y^2 must be a multiple of $(a-b=x-a)$. Now if b is greater than $(x-a)$, we can let $(x-a)=(b-c)$ and divide as before, thus repeating the same process until the first base root of the remainder becomes less than the factor $(x-a)$. At that point, if the last base root of the remainder is greater than the factor $(x-a)$, we may reverse the remainder and divide in the same manner as before, until both base roots of the remainder become

less than $(x-a)$. That is, if b is less than $(x-a)$ and y is is greater than $(x-a)$, we may let $(y-c)=(x-a)$, and reverse the remainder (b^2-by+y^2) and divide,

$$(y^2-by+b^2)\div(y-c)=y-b+c+\frac{c^2-bc+b^2}{y-c}, \text{ etc., etc.}$$

Now, if c^2-bc+b^2 is not equal to $[(x-a)=(y-c)]$, and both c and b are less than $(x-a)$ we cannot divide any further by a factor equal to $(x-a)$. Never the less, $(x-a)$ is still a factor of this last remainder (c^2-bc+b^2), but being greater than either of the base roots, c or b, it must be the greater of the two factors of (c^2-bc+b^2), and the other factor must be smaller than the greater of c or b. Now suppose b is greater than c, let the smaller factor be $(b-d)$ and divide

$$(b^2-bc+c^2)\div(b-d)=b-c+\frac{d^2-cd+c^2,}{b-d} \text{ in which}$$

$$b-c+d+\frac{d^2-cd+c^2}{b-d}=(x-a).$$

By repeating the various foregoing processes we must eventually come to a point where the remainder will be equal to the divisor, because with each division the remainder approaches unity* as a limit at which point, if not before, it must equal the divisor. At the point where the dividing factor and the remainder become equal the dividing factor is shown to be expressible as a frustum in the form of the remainder.

From this point, by the application of proposition 10, we may work back through the entire process and show that each divisor and quotient, in turn, may be expressed as a frustum.

*It may be shown that the remainder will equal the divisor before becoming less than three, but I do not consider that necessary to the above.

Proposition 21, Theorem. The six sets of base roots composing a cycle express only three different sets of bases.

Given; the six sets of base roots composing a cycle, as per proposition 11.

(1.) x^2+xy+y^2
(2.) $(x+y)^2+(x+y)(-x)+(-x)^2$
(3.) $y^2+y[-(x+y)]+[-(x+y)]^2$
(4.) $(-x)^2+(-x)(-y)+(-y)^2$
(5.) $[-(x+y)]^2+[-(x+y)]x+x^2$
(6.) $(-y)^2+(-y)(x+y)+(x+y)^2$

Now (1) and (4), when cleared of parentheses, each equal x^2+xy+y^2

(2) and (5) each equal $(x+y)^2-(x+y)x+x^2$

(3) and (6) equal $y^2-y(x+y)+(x+y)^2$

Hence, the six different sets of base roots express only three different sets of bases.

Proposition 22, Theorem. In the three sets of bases expressed by a cycle of relatively prime base roots, one and only one set may be expressed by base roots which are both odd integers.

Given; the three sets of bases expressed by a cycle of base roots as per proposition 21.

(1.) x^2+xy+y^2
(2.) $(x+y)^2-(x+y)x+x^2$
(3.) $y^2-y(x+y)+(x+y)^2$

in which x and y are relatively prime integers.

Prove that in one and only one frustum the base roots are both odd integers.

Proof; x and y, being relatively prime, are either both odd, or one odd and one even. Now, if both are odd, both base roots of (1) are odd, but the first base root of (2)

and the last base root of (3) are both even. However, if x is odd and y is even, both base roots of (2) are odd, but the last base root of (1) and the first base root of (3) are both even. Likewise, if y is odd and x is even, both base roots of (3) are odd, but the first base root of (1) and the last base root of (2) are both even. Hence, in any case, the base roots of one and only one are both odd.

Proposition 23, Problem. To find an arrangement of base roots for two frustums whose product will express, in reversed order, the base roots of the product of two given frustums.

Given; two frustums (a^2+ab+b^2) and (x^2+xy+y^2) whose product is

$$(ay+bx+by)^2+(ay+bx+by)\,(ax-by)+(ax-by)^2.$$

To find an arrangement of base roots whose product will express that product with its base roots reversed. Write the one frustum with its bases in reversed order, and write the other frustum in the reversed order of the arrangement which precedes the other frustum in its standard cycle.

Example;

$$(b^2+ba+a^2)\times[(x+y)^2+(x+y)\,(-y)+(-y)^2] \text{ equals}$$

$$(ax-by)^2+(ax-by)\,(ay+bx+by)+(ay+bx+by)^2,$$

which is equal to the reversed arrangement of base roots of the product of the two given frustums.

Proposition 24, Theorem. If members of the same cycle be multiplied by a given frustum without reversing the order of any of the base roots, their products will all be members of the same cycle.

Given; all the members of any cycle;

(1.) $x^2 \quad +xy \quad +y^2,$

(2.) $(x+y)^2+(x+y)\,(-x)+(-x)^2,$

(3.) $y^2+(y)[-(x+y)+[-(x+y)]^2$,
(4.) $(-x)^2+(-x)(-y)+(-y)^2$,
(5.) $[-(x+y)]^2+[-(x+y)]x+x^2$,
(6.) $(-y)^2+(-y)(x+y)+(x+y)^2$,

and the frustum (a^2+ab+b^2), to be multiplied by each member of the given cycle.

To show that all the products will be members of the same cycle, multiply each member of the given cycle. separately, by the given frustum and compare the products.

$(a^2+ab+b^2)\times$ 1st member=
$$(ay+bx+by)^2+(ay+bx+by)(ax-by)+(ax-by)^2.$$

$(a^2+ab+b^2)\times$ 2nd member=
$$[-(ax-by)]^2+[-(ax-by)](ax+ay+bx)+(ax+ay+bx)^2.$$

$(a^2+ab+b^2)\times$ 3rd member=
$$[-(ax+ay+bx)]^2+[-(ax+ay+bx)](ay+bx+by)+(ay+bx+by)^2.$$

$(a^2+ab+b^2)\times$ 4th member=
$$[-(ay+bx+by)]^2+[-(ay+bx+by)][-(ax-by)]+[-(ax-by)]^2.$$

$(a^2+ab+b^2)\times$ 5th member=
$$(ax-by)^2+(ax-by)[-(ax+ay+bx)]+[-(ax+ay+bx)]^2.$$

$(a^2+ab+b^2)\times$6th member=
$$(ax+ay+bx)^2+(ax+ay+bx)[-(ay+bx+by)]+[-(ay+bx+by)]^2.$$

Now, in comparing the above products it will be observed that if to any one of the products we apply the formula for finding the next succeeding member of a cycle as per proposition 11, the result will be equal to the previous product listed above. Hence the products are all members of the same cycle, but related to each other in an order reversed to that of a standard cycle.

Proposition 25, Theorem. If two frustums be multiplied together, and then each frustum reversed and multiplied together again, the second product will be a member of the cycle of the first product written in reversed order.

Given; two frustums, (a^2+ab+b^2) and (x^2+xy+y^2), to be multiplied together, first as $(a^2+ab+b^2)\times(x^2+xy+y^2)$, and then as $(b^2+ba+a^2)\times(y^2+yx+x^2)$. To show that the last product is a member of the cycle of the first product written in reversed order.

Proof; $(a^2+ab+b^2)\times(x^2+xy+y^2)=$

$$(ay+bx+by)^2+(ay+bx+by)(ax-by)+(ax-by)^2$$

and $(b^2+ba+a^2)\times(y^2+yx+x^2)=$

$$(bx+ay+ax)^2+(bx+ay+ax)(by-ax)+(by-ax)^2$$

Now, if the first product be called the 1st member of its cycle, then the last product will be equal to the 6th member of that same cycle, but written in reversed order.

Proposition 26, Theorem. If a frustum in which the base roots have a common divisor, be multiplied by another frustum, the base roots of the product will both contain that same common divisor.

Given; a frustum (a^2+ab+b^2) in which a, and b, are each divisible by the common divisor d, so that $a=md$, and $b=nd$.

Now (a^2+ab+b^2) may be written

$$(md)^2+(md)(nd)+(nd)^2.$$

Multiplying that by another frustum x^2+xy+y^2, as per proposition 3, will give the product $(mdy+ndx+ndy)^2+(mdy+ndx+ndy)(mdx-ndy)+(mdx-ndy)^2$, each term of which is divisible by d, the common divisor of a, and b.

Proposition 27, Theorem. If members of two different cycles, expressing the same frustum, be multiplied together, the base roots of their product will have a common divisor.

Given; $(a^2+ab+b^2)=(x^2+xy+y^2)$, members of different cycles expressing the same frustum.

Let (p^2+pq+q^2) and (r^2+rs+s^2) equal their product as per proposition 4.

To prove that p and q, or r and s, have common divisors. Let (m^2+mn+n^2) and (t^2+tu+u^2), be the factors into which x^2+xy+y^2, and a^2+ab+b^2, may be resolved, as per proposition 6, in such a manner that $a=(tn+um+un)$, $b=(tm-un)$, $x=(tm+um+un)$, and $y=(tn-um)$. Now, let $(p^2+pq+q^2)=(a^2+ab+b^2)\times(x^2+xy+y^2)$ multiplied as per proposition 3, so that $p=(ay+bx+by)$, and $q=(ax+by)$. Substituting the above values of a, b, x, and y, so as to express the value of p and q in terms of m, n, t, and u. $p=(t^2-u^2)\times(m^2+mn+n^2)$, $q=(2tu+u^2)\times(m^2+mn+n^2)$.

In the same manner, let $(r^2+rs+s^2)=(a^2+ab+b^2)\times(y^2+yx+x^2)$ so that $r=(ax+by+bx)$, and $s=(ay-bx)$. Substituting as above,

$$r=(2mn+m^2)\times(t^2+tu+u^2), \qquad s=(n^2-m^2)\times(t^2+tu+u^2).$$

Hence, p and q, have a common divisor (m^2+mn+n^2), and r and s, have a common divisor (t^2+tu+u^2).

Proposition 28, Theorem. An identical arrangement of the base roots of the product of any three frustums taken together in a given order, may be obtained by multiplying the product of either of the first two frustums, taken in regular order, and the last frustum reversed, by the other of the first two frustums reversed.

Given; three frustums

(a^2+ab+b^2), (m^2+mn+n^2), and (x^2+xy+y^2).

To show that the product of

$[(a^2+ab+b^2)\times(m^2+mn+n^2)]\times(x^2+xy+y^2)$ is exactly the same as $[(a^2+ab+b^2)\times(y^2+yx+x^2)]\times(n^2+nm+m^2)$ or exactly the same as

$$[(m^2+mn+n^2)\times(y^2+yx+x^2)]\times(b^2+ba+a^2).$$

Multiplying each arrangement out, as per proposition 3, and each product will be found to be, $(amx-bnx+any+bmy+amy)^2+(amx-bnx+any+bmy+amy)\times(anx+bmx+bnx-amy+bny)+(anx+bmx+bnx-amy+bny)^2$, thus being exactly the same.

Proposition 29, Theorem. If the cube of a frustum be computed as the product of one member each of three different cycles, each of which expresses that frustum, the cube will be a frustum whose base roots contain a common divisor.

Given; $(a^2+ab+b^2)=(m^2+mn+n^2)=(x^2+xy+y^2)$ each a member of a different cycle, but expressing the same frustum.*

To prove that their product, taken in any order, will be a frustom whose base roots contain a common divisor.

Proof; whichever two sets be multiplied together first, their product will be a frustum whose base roots have a common divisor (proposition 27), and that product multiplied by the third set will likewise be a frustum whose base roots have a common divison, (proposition 26).

Proposition 30, Theorem. If the cube of a frustum be computed as the product of two members of one cycle, and one member of another cycle, each of which express that frustum, the cube will be a frustum whose base roots contain a common divisor.

Given; (a^2+ab+b^2) and $(a_1^2+a_1b_1+b_1^2)$, members

*NOTE:—Regardless of which members of the given cycles be used as multiples, the products will all be in the same cycle, (proposition 24) and consequently each set of base roots will have the same common divisor (proposition 12).

of the same cycle and (x^2+xy+y^2) member of another cycle but equal to (a^2+ab+b^2).

To show that the prodoct of the three given expressions, is a frustum whose bases have a common divisor.

First; If they be multiplied in the order

$$[(a^2+ab+b^2)\times(x^2+xy+y^2)]\times(a_1{}^2+a_1b_1+b_1{}^2),$$

the product of the first two expressions would be a frustum whose bases have a common divisior (proposition 27), which when multiplied by the last expression would also give a frustum whose bases have a common divisor, (proposition 26).

Second; If they be multiplied in the order

$$[(a^2+ab+b^2)\times(a_1{}^2+a_1b_1+b_1{}^2)]\times(x^2+xy+y^2),$$

their product would be identical with the product of the arrangement

$$[(a^2+ab+b^2)\times(y^2+yx+x^2)]\times(b_1{}^2+b_1a_1+a_1{}^2),$$

(proposition 28), and, as in the first case, their product would be a frustum whose bases have a common divisor (proposition 27), and (proposition 26).

Proposition 31, Theorem. If the cube of a frustum be computed from only one cycle of base roots, that cube may be expressed by only one cycle of relatively prime bases.

Proof; Multiply any frustum (x^2+xy+y^2) by itself, in the following orders; $[(x^2+xy+y^2)\times(x^2+xy+y^2)]$ and $[(x^2+xy+y^2)\times(y^2+yx+x^2)]$ making two different expressions of its square as per (proposition 4.)

Then multiply each of these two expressions of the square of the frustum, by the frustum again, first in its regular order, and then with it reversed, making the four following orders of multiplication, and all possible cycles expressing the cube;

(1st) $[(x^2+xy+y^2)\times(x^2+xy+y^2)]\times(x^2+xy+y^2)$
(2nd) $[(x^2+xy+y^2)\times(y^2+yx+x^2)]\times(x^2+xy+y^2)$
(3rd) $[(x^2+xy+y^2)\times(x^2+xy+y^2)]\times(y^2+yx+x^2)$
(4th) $[(x^2+xy+y^2)\times(y^2+yx+x^2)]\times(y^2+yx+x^2)$.

Now, the base roots of the 1st, 2nd, and 4th arrangements, when multiplied out, will each be divisible by the original frustum (theorem 9), and will therefore not be relatively prime, but the 3rd arrangement, when multiplied out, becomes;

$$(x^3+3x^2y-y^3)^2+[(x^3+3x^2y-y^3)\times(-x^3+3xy^2+y^3)]+(-x^3+3xy^2+y^3)^2,$$

a frustum whose bases may easily be shown to be relatively prime. Hence, the cycle of which this frustum is a member is the only cycle which may express the cube of the original frustum, as a frusum with relatively prime bases.

See also note on proposition 29, and converse of proposition 12.

Proposition 32, Theorem. The cube of any frustum may be expressed as a frustum whose bases are relatively prime, and both odd integers, in only one form.

Proof; The cube of a frustum may be computed in only three different ways.*

1st. By the product of one member each of three different cycles.

2nd. By the product of any member of one cycle taken twice, and a member of another cycle take once.

3rd. By the product of any member of one cycle taken three times.

*NOTE:—This is on the assumption that all the factors of a composite frustum have already been combined so that we have at hand all the different sets and cycles of base roots by which the given frustum may be expressed, as per proposition 19. From this point there are possible only three different manners, those shown above, in which any of these sets of base roots may be combined to produce the cube of the frustum. Hence any expression of the cube of the frustum may be produced in one of these manners, for (proposition 18).

But the product of the 1st combination gives a frustum whose bases have a common divisor (Theorem 29). The same is true of the second combination, (Theorem 30). In the 3rd combination only one of four possible cycles is composed of sets of relatively prime base roots, (Theorem 31). And in any cycle there may be one and only one frustum whose bases are both odd integers, (Theorem 22). Hence there is only one form in which the cube of a frustum may be expressed as a frustum whose bases are relatively prime and both odd integers, and that is as follows;

The cube of (a^2+ab+b^2) expressed as a frustum, as per proposition 31, is

$$[(a^3+3a^2b-b^3)^2+(a^3+3a^2b-b^3)(-a^3+3ab^2+b^3)+(-a^3+3ab^2+b^3)^2].$$

Proposition 33, Theorem. The difference between the cubes of two integers cannot equal the cube of an integer.

To prove that $x^3-y^3=z^3$ cannot be satisfied by whole numbers, Let x, y, and z, be the smallest integers which may satisfy that equation. Also let x and y be relatively prime and both odd. Then prove that $(x-y)\times(x^2+xy+y^2)$ is not a cube.

First; Let $(x-y)$ be not divisible by 3, then $(x-y)$ and (x^2+xy+y^2) are relatively prime, so that each is a cube. Now, since (x^2+xy+y^2) is a cube, it must be the cube of a number (a^2+ab+b^2) of like form, (theorem 20.) But

$$[(a^3+3a^2b-b^3)^2+(a^3+3a^2b-b^3)(-a^3+3ab^2+b^3)+(-a^3+3ab^2+b^3)^2]$$

is the only expression of the cube of (a^2+ab+b^2) whose base roots are relatively prime and both odd, (theorem 32), Hence that must be identically (x^2+xy+y^2), and $(x-y)$ then equals $(2a^3+3a^2b-3ab^2-2b^3)$, whose factors are $(a-b)$, $(2a+b)$, and $(a+2b)$. But $(x-y)$ is a cube, therefore its factors, which are relatively prime, are each a cube, but the second minus the last equals the first, Hence $(a-b)$,

a cube, equals the difference between two cubes. Thus we have two cubes much smaller than x^3, and y^3, whose difference is a cube, but that is contrary to our assumption that x, y, and z, are the smallest integers which will satisfy that condition.

Now; Let $(x-y)$ be divisible by 3. Then (x^2+xy+y^2) is a multiple of 3, but not a multiple of 9, (theorem 2). But $(x-y)(x^2+xy+y^2)=z^3$ is a multiple of 27, hence $(x-y)$ must be a multiple of 9. Then $3(x-y)$, and $\dfrac{x^2+xy+y^2}{3}$ are relatively prime and each a cube. Now, the number 3, being a frustum $(1^2+1\times1+1^2)$, it follows that $\dfrac{x^2+xy+y^2}{3}$ must also be a frustum, (theorem 20), and since it is also a cube it must be the cube of a number (m^2+mn+n^2) of like form (theorem 20.) But the cube of (m^2+mn+n^2) as per theorem 32, multiplied by the frustum 3, written $[(-1)^2+(-1)(2)+2^2]$; as per proposition 3, equals

$$[(m^3+6m^2n+3mn^2-n^3)^2+(m^3+6m^2n+3mn^2-n^3)$$
$$(m^3-3m^2n-6mn^2-n^3)+(m^3-3m^2n-6mn^2-n^3)^2],$$

the only expression of $3(m^2+mn+n^2)^3$ written as a frustum whose bases are odd relative prime integers, (theorem 32), Hence it must be identically, (x^2+xy+y^2), and $(x-y)$ then equals $(9m^2n+9mn^2)$, or the cube $3(x-y)$, equals $(27m^2n+27mn^2)$. Therefore the relatively prime factors 27, $m+n$, m, and n, must each be a cube. But the sum of the last two factors equals the second factor. Thus, we have two cubes, m, and n, much smaller than y^3 and z^3 whose sum equals a cube, which, as in the first case, is contrary to our assumption that x, y, and z, are the smallest integers which will satisfy that condition.

EULER'S PROOF OF THE PROBLEM OF THE CUBES

Let x and y be relatively prime, both odd, and the smallest of such numbers which will satisfy the equation. $x^3+y^3=z^3$.

Let $x+y=2p$, and $x-y=2q$, We have now to prove that $2p(p^2+3q^2)$ is not a cube.

First, suppose it is a cube, and that p is not a multiple of 3.

Then $\frac{p}{4}$, and p^2+3q^2 are relatively prime, so that each is a cube.

*Since p^2+3q^2 is a cube, it must be the cube of t^2+3u^2 of like form.

*And $p+q\sqrt{-3}$ is the cube of $t+u\sqrt{-3}$.

Hence $p=t(t^2-9u^2)$, and $q=3u(t^2-u^2)$.

Since $\frac{p}{4}$ is a cube, $2p$ must also be a cube.

Hence $2t$, $t+3u$, and $t-3u$, must each be cubes, and relative prime, since p and t are not multiples of 3.

Now if the last two of the above factors are cubes, a^3 and b^3, we then have two cubes a^3 and b^3 much smaller than x^3 and y^3, whose sum is a cube $2t$.

Now in the case where p is a multiple of 3, let $p=3r$.

Then the product of the relatively prime numbers $\frac{9r}{4}$ and $3r^2+q^2$ is a cube.

Now $r=3u(t^2-u^2)$, and since $\left(\frac{8}{27}\right)\left(\frac{9r}{4}\right)=\frac{2r}{3}=$ $2u(t+u)(t-u)$ is a cube and is the product of three relatively prime factors, each factor is a cube. $t+u=a^3$, $t-u=b^3$, so that a^3-b^3 is a cube $2u$, and as above, much smaller than x^3 and y^3.

*Assumed to be true, but Euler was unable to give a rigorous proof of the same.

COMPLETION OF EULER'S PROOF

1st. To prove his lemma that since p^2+3q^2 is a cube, it is a cube of a number t^2+3u^2 of like form.

Let, as he did, $x+y=2p$, and $x-y=2q$, x and y relatively prime and both odd.

Then $p^2+3q^2=x^2-xy+y^2$.

Now let $a+b=2t$, and $a-b=2u$, Then $t^2+3u^2=a^2-ab+b^2$, But if x^2-xy+y^2 is a cube it is a cube of a number a^2-ab+b^2 of like form (theorem 20). (Every factor of a composit frustum may be expressed as a frustum) Therefore p^2+3q^2 is the cube of a number t^2+3u^2 of like form.

2nd. His critics were right in stating that Euler's $t+u\sqrt{-3}$ might have a cube other than $p+q\sqrt{-3}$, but $p+q\sqrt{-3}$ is the only cube of $t+u\sqrt{-3}$ which can satisfy the conditions that x and y are relative prime and both odd, when $p=t(t^2-9u^2)$ and $q=3u(t^2-u^2)$.

Let, as in the proof of the 1st lemma, $t^2+3u^2=a^2-ab+b^2$, Then the only form in which its cube may be expressed as a frustum whose bases are relatively prime and both odd is the form as per the 3rd product in Theorem 31, (see Theorem 32), which is as follows

$$(-a^3+3a^2b-b^3)^2-(-a^3+3a^2b-b^3)(-a^3+3ab^2-b^3)+(-a^3+3ab^2-b^3)^2,$$

this being as in the 1st lemma identically x^2-xy+y^2, it follows that $x=(-a^3+3a^2b-b^3)$ and $y=(-a^3+3ab^2-b^3)$, Thus $2p=x+y=(-2a^3+3a^2b+3ab^2-2b^3)$, and $2q=x-y=(3a^2b-3ab^2)$.

but $$(-2a^3+3a^2b+3ab^2-2b^3)=\frac{(a+b)^3-9(a+b)(a-b)^2}{4}=2(t^3-9tu^2).$$

Hence, $2p=2(t^3-9tu^2)$, $p=t(t^2-9u^2)$.

$$2q=3a^2b-3ab^2=\frac{3(a+b)^2(a-b)}{4}-\frac{3(a-b)^3}{4}=2(3t^2u-3u^3),$$

$$q=3u(t^2-u^2).$$

Therefore $p=t(t^2-9u^2)$ and $q=3u(t^2-u^2)$ are the only values of p and q which will make x and y both odd and relatively prime, but when $p=t(t^2-9u^2)$ and $q=3u(t^2-u^2)$, $p+q\sqrt{-3}$ is the cube of $t+u\sqrt{-3}$, hence $p+q\sqrt{-3}$ is the only cube of $t+u\sqrt{-3}$ which will satisfy those conditions of x and y.

Therefore Euler's lemmas are correct in every detail.